ON WEAKLY F(X) - CLEAN SEMIRING

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Abstract

A Semiring R With identity is Called "clean semiring" if for every element $r \in R$, there exist an idempotent 'e' and 'a' unit I in R such that $r = i + e$. Let *C(R)* denote the center of a semiring R and f(x) be a polynomial in C(R) [x]. An element $a \in R$ is called '$f(x) - clean$ " if $a = u + s$ where '$f(s)=0$ and 'u' is a unit of R and R is *f(x) –clean* if where every element is *f(x) –clean*. In this paper we define a semiring to be a weakly *f(x)-clean* if each element of R can be written as either the sum or difference of a unit and a root of *g(x).*

Keywords :- Clean semiring , Weakly f(x) clean semiring, Strongly clean semiring.

1. Introduction

Throughout this paper, R is an associative semiring with identity . A semiring R is called clean semiring if for every element $a \in R$, there exist an idempotent 'e' and 'a' unit I in R such that $r = i$+e and R is called strongly clean semiring if in addition, $eu = ue$. Let C(R) denote the center of a semiring R and *f(x)* be a polynomial in C (R)[x]. Following camillo and simon , an element $r \in R$ is called $f(x) - clean$ if $r = x + s$ where $f(s)= 0$ and u is a unit of R and R is *f(x)- clean* if every element in R is $f(x) - clean$. It is clear that the (x^2 -x) – clean semirings are precisely the clean semirings.

Definition 2.1: A semiring R is called clean semiring if for every $a\epsilon$ R there exist an idempotent ' e' and a unit u in R such that $a = e + u$.

Definition 2.2: A semiring R is called strongly clean if for every $a\epsilon R$ there exist an idempotent 'e' and a unit u in R such that $a = e + u$ and $eu = ue$.

Definition 2.3: Let f(x) be a fixed polynomial in C(R)[x]. An element $r\ \epsilon R$ is called weakly *f(x)-clean* semiring if $r = u + s$ *or* $r = u - s$ where g(s) =0 and $u \in U(R)$. We say that R is weakly f(x)-clean semiring if every element is weakly *f(x)- clean* semirng.

Proposition 2.4: Let g: R→ S be a semiring epimorphism. If R is weakly *f(x)-clean* then S is weakly *f(x)-clean*.

Proof: Let *f(x)* be a n^{th} degree polynomial.

f(x) = $a_0 + a_1x + a_2x^2 +$ ....$a_nx^n \in C(R)[x]$. Then g(f(x)) = g($a_0 + a_1x + a_2x^2 +$ ....$a_nx^n) \in C(s)[x]$. As $s \in S$ there exist $u \in U(R)$ and $s_0 \in R$ such that $r = u \pm s_0$ and f(s_0) = 0. Then $S = g(r) = g(u \pm s_0) = g(u) \pm g(s_0)$; $g(u) \in U(S)$.

But $g'\left(f\left(g(s_0)\right)\right) = g(a_0) + g(a_1)g(s_0) + \cdots g(a^n)g(s_0^n)$.

 DOI : 10.36893.DRSR.2021.V08I14.70-75

$= g\,(a_0 + a_1 s_0 + a_2 {s_0}^2 + \ldots\ldots\ldots\ldots a_n {s_0}^n)$

$= g\,(f\,(s_0))$

$= g\,(0)$

$= 0.$

We have S is weakly $g'(\mathrm{f(x)})$ – clean.Therefore S is weakly $g'(f(x))$- clean semiring.

Proposition 2.5: Let f(x) ∈ Z(x) and $\{R_i\}i \in I$ be a family of semirings .Then $\pi_{i\in I}\, R_i$ is weakly *f(x)*-clean semiring if and only if for all $i \in I, R_i$ is weakly *f(x)*- clean semiring.

Proof: Let us define a mapping $\pi_j : \pi_{i\in I}\, R_i \to R_j$ by $\pi_j(a_i)i \in I) = a_j \forall j \in I$. π_j is a smiring epimorphism for every $i \in I$, each R_i is a weakly f(x)- clean semiring.

For the converse, Let $x = \{x_i\}i \in I \in R = \pi_{i\in I} R_i$. In R_{i0}, We can write, $x_i = u_{i0} + s_{i0}$ or $x_i = u_{i0} - s_{i0}$ where $u_{i0} \in U(R_{i0})$ and $g(s_{i0})$= 0. If $x_{i0} = u_{i0} + s_{i0}$ for $i \neq i_0$, Let $x_i = u_i + s_i$ where $u_i \in U(R_i), g(s_i)$= 0 while if $x_{i0} = u_{i0} - s$ for $i \neq i_0$. Let $x_i = u_i - s_i$ where $u_i \in U(R_i), g(s_i)$= 0.

Then $u = \{u_i\}i \in I$ and $g(s) = \{s_i\}i \in I$

$= a_0\{1\}_{i\in I} + a_1\{s_i\}_{i\in I} + \ldots\ldots\ldots + a_n\{s_i^n{}_i\}_{i\in I}$

$= \{a_0\}_{i\in I} + \{a_1 s_i\}_{i\in I} + \ldots\ldots\ldots + \{a_n s_i^n{}_i\}_{i\in I}$

$= \{a_0 + a_1 s_i + \ldots\ldots\ldots + a_n s_i^n{}_i\}_{i\in I}.$

$= \{f\,(s_i)\}i \in I.$

$= 0.$

That is $\pi_{i\in I}\, R_i$ is wealy *f(x)* – clean semiring.

Theorem 2.6: Let R be a semiring f(x)∈ $C(R)[x]$ and $n \in N$.Then R is weakly f(x)- clean if and only if the upper triangular matrix semiring$T_n(R)$ is weakly *f(x)* –clean.

Proof: Let R be a weakly *f(x)*-clean and A = $(a_{ij}) \in T_n(R)$ with $a_{ij} = 0$ for $i \le j \le n$. Since R is weakly f(x)- clean for any $1 \le i \le n$. Then there exist $s_{ii} \in R$ and $u_{ii} \in U(R)$. Such that $a_{ii} = u_{ii} \pm s_{ii}$ with $g(s_{ii})$= 0. So we have.

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ 0 & a_{22} & .. & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ..... & a_{nn} \end{bmatrix} = \begin{bmatrix} u_{11} \pm s_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ 0 & u_{22} \pm s_{22} & \ldots\ldots\ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots\ldots\ldots & u_{nn} \pm s_{nn} \end{bmatrix}$$

In R for any $0 \le i \le n$,We can write $a_{ii} = u_{ii} + s_{ii}$or $a_{ii} = u_{ii} - s_{ii}$where $u_{ii} \in U(R), g(s_{ii})$= 0. If $a_{ii} = u_{ii} + s_{ii}$for $j \neq i$. Let $a_{jj} = u_{jj} + s_{jj}$ where $u_{jj} \in U(R)$, $g(s_{jj})$= 0. While if $a_{ii} = u_{ii} - s_{ii}$ for $j \neq i$. Let $a_{jj} = u_{jj} - s_{jj}$ where $u_{jj} \in U(R), g(s_{jj})$= 0. Then by the elementary row and column operations we can see that

 DOI : 10.36893.DRSR.2021.V08I14.70-75

$$U = \begin{bmatrix} u_{11} & a_{12} & a_{13}\dots\dots\dots & a_{1n} \\ 0 & u_{22} & a_{23}\dots\dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0\dots\dots\dots & u_{nn} \end{bmatrix} \in GL_n(R)$$

Suppose $g(x) = \sum_{i=0}^{m} a_i x^i \in c(R)[x]$ then

$$g(s = \begin{bmatrix} s_{11} & 0 & .. & 0 \\ 0 & s_{22} & .. & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ..... & s_{nn} \end{bmatrix}) = a_0 I_n + a_1 s + a_2 s^2 + \dots\dots + a_n s^n$$

$$= \begin{bmatrix} a_0 & 0 & .. & 0 \\ 0 & a_0 & .. & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ..... & a_0 \end{bmatrix} + \begin{bmatrix} a_1 s_{11} & 0 & .. & 0 \\ 0 & a_1 s_{22} & .. & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ..... & a_1 s_{nn} \end{bmatrix} + \dots\dots\dots\dots\dots\dots + \begin{bmatrix} a_m s_{11}{}^m & 0 & .. & 0 \\ 0 & a_m s_{22}{}^m & .. & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ..... & a_m s_{nn}{}^m \end{bmatrix}$$

$$= \begin{bmatrix} g(s_{11}) & 0 & .. & 0 \\ 0 & g(s_{22}) & .. & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ..... & g(s_{nn}) \end{bmatrix} = 0$$

So, $T_n(R)$ is weakly f(x) – clean .

Now let $T_n(R)$ be weakly f(x)- clean.

Define $\theta : T_n(R) \to R$ by $\theta(A) = a_{11}$

DOI : 10.36893.DRSR.2021.V08I14.70-75

Where $A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ 0 & a_{22} & .. & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ..... & a_{nn} \end{bmatrix}$. Then θ is a semiring epimorphsm . For any $a \in R$, let B be the diagonal matrix diag(a_{11}, a_{22}, ……..a_{nn}). Then a = θ(B) = θ(U±S) = θ(U) ± θ(S) where U ∈ $GL_n(R)$ and

$g(\theta(S) = a_0 + a_1\theta(S) + \ldots\ldots + a_n\theta(S^n)$

$= \theta(B_0) + \theta(B_1)\theta(S) + \ldots\ldots + \theta(B_n)\,\theta(S^n)$

$= \theta(B_0 + B_1 S + \ldots\ldots + B_n S^n)$

$= \theta(a_0 I_n + (a_1 I_n)S + \ldots\ldots + (a_n I_n)S^n)$

$= \theta(g(S))$

$= 0$

Thus a is weakly f(x)- clean i.e R is a weakly f(x) –clean semiring.

Preposition 2.7: Let R be a semiring and g(x) ∈ C(R)[x].Then the formal power series . Semiring R[[t]] is weakly f(x) –clean semiring if and only if R is weakly f(x)- clean.

Proof: Let R be weakly f(x)- clean and $g = \sum_{i\geq 0} a_i t^i \in R[[t]]$. Since R is weakly f(x)-clean $a_0 = u \pm s$ for some s∈ R, u∈ U(R) and g(S) = 0. Then f = $(u + \sum_{i\geq 1} a_t t^i) \pm S$, $u + \sum_{i\geq 1} a_t t^i \in U(R[[t]]$. So g is weakly f(x) –clean i.e. *R[t]* be weakly f(x) –clean.For the converse, let *R[[t]]* be weakly f(x) –clean. Since $\theta: R[[t]] \rightarrow R$ with $\theta(f) = a_0$ is a semiring epimorphism. Where g = $\sum_{i\geq 1} a_t t^i \in R[[t]]$. *R* is weakly *f(x)- clean* semiring.

Remark 2.8: Generally, the polynomial semiring R[t] is not weakly f(x)- Clean semiring for non-zero polynomial f(x) ∈ C(R)[x]. For example , Let *R* be a commutative semiring and also let *f(x)=x*. We show that *t* is not weakly f(x)- clean. If t = u $\pm s$ then it must be that S=0 and so t = u. clearly t $\notin U(R[U])$. Therefore R [t] is not weakly.

Theorem 2.9: Let R be a commutative semiring , M an R-module and g(x)= $\sum_{i=0}^{n} a_i x^i \in R[x]$. If R is a weakly f(x) –clean semiring, then the idealization R(M) of R and M is weakly f(x) – clean semiring.

Proof: Let (r,m) ∈ R(M). Since R is a weakly f(x)- clean semiring, we have r=u±s where u ∈ U(R) and g(s)=0.So (r,m)=(u±s, m)=(u,m)±(s,0).we have (u,m)$(u^{-1}, -u^{-1}mu^{-1})$=(uu^{-1}, $u(-u^{-1}mu^{-1})$+mu^{-1})=(1,0).

Therefore (u,m)∈ U(R(M)).Also we have g((s,0))= $a_0(1,0) + a_1(s,0) + \ldots\ldots +) + a_n(s,0)^n$

$= a_0(1,0) + a_1(s,0) + \ldots\ldots +) + a_n(s^n, 0)$

$= (a_0, 0) + (a_1 s, 0) + \ldots\ldots +) + a_n(s^n, 0)$

$= (a_0 + a_1 s + \ldots\ldots + + a_n s^n, 0)$

= (g(s)), 0)

= (0, 0)

Thus (r, m) is weakly f(x) –clean and so R(M) is a weakly f(x)- clean semiring.

3. Weakly (x^n-x)- Clean semirings.

In this section we consider the weakly ($x^n - x$)-clean semirings.

Theorem 3.1: Let R be a semiring , n∈ N and a,b ∈ R. Then R is weakly ($ax^{2n} - bx$)-clean if and only if R is weakly ($ax^{2n} + bx$)-clean.

Proof: Suppose R is weakly ($ax^{2n} - bx$) – clean semiring. Then for any $r \in R$, $-r = u \pm s$ where ($as^{2n} - bs$)=0 and u ∈ U(R). So $r=(-u)\pm(-s)$ where $(-u) \in U(R)$ and $a(-s)^{-2n} + b(-s) = 0$. Hence r is weakly ($ax^{2n} + bx$)-cleansemiring. Therefore, R is weakly C($ax^{2n} + bx$)-cleansemiring. Now suppose R is weakly ($ax^{2n} + bx$)-clean. Let $r \in R$. Then there exist s and u such that $-r = u \pm s$, $as^{2n} + bs = 0$ and u ∈ U(R). So $r = (-u) \pm -(s)$ satisfies $as^{2n} - bs = 0$.

Hence, R is weakly ($ax^{2n} - bx$)-clean semiring.

Proposition 3.2: Let R be a weakly $(x^n - x)-$clean semiring where n≥2 and a∈ R.Then either (i) $a=u \pm v$ where u∈ U(R) and $v^{n-1}=1$ (ii) both aR and Ra contain nontrivial idempotents.

Proof: Since R is weakly $(x^n - x)-$clean semiring, write $a=u+e$ where u∈ U(R) and $e^n=e$. Then $ae^{n-1}=ue^{n-1} \pm e$. So $a(1-e^{n-1}) = u(1-e^{n-1})$. Since $a(1-e^{n-1})$ is an idempotent. $u(1-e^{n-1}) = fw$ where □ ∈ U(R) and $f^2=f$∈ □. So $f = a(1-e^{n-1})w^{-1} \in aR$.

Suppose (i) does not hold. Then $(1-e^{n-1}) \neq 0$, hence $f \neq 0$. Thus aR contains a nontrivial idempotent. Similarly, Ra contains a nontrivial idempotent.

Definition 3.3: An element r∈ □ is called weakly n-clean if $r = u_1 + u_2 + ...+ u_n \pm e$ with $e^2 = e \in R$ and $u_i \in U(R)$ $for 1 \leq i \leq n$ and R is called weakly n-clean if every element of R is weakly n-clean semiring.

Definition 3.4: An element a∈ □ is called right π- regular if it satisfies the following equivalent conditions.

i. $a^n \in a^{n+1}R$ for some integer n≥1;
ii. $a^n R = a^{n+1}R$ for some integer n≥1;
iii. The chain $aR \supseteq a^2R \supseteq$....terminates. The left π-regular elements are defined analogously. An element a∈ R is called strongly π-regular if it is both left and right π-regularand R is called strongly π-regular if every element is strongly π-regular.

Proposition 3.5: Let n ∈N, if the ring R is weakly $(x^n - x)-$clean semiring, then R is weakly 2-clean semiring.

Proof: Let r ∈ R. Then r = u±□ for some $t^n = t \in R$ and u ∈ U(R). Since t is a strongly π-regular element and strongly π-regular elements are strongly clean semirings, t = v ± e for some $e^2 = e \in R$ and u ∈ U(R). Then $r = u \pm v \pm e$ is weakly 2-clean semiring. Hence R is weakly-2 clean semiring.

 DOI : 10.36893.DRSR.2021.V08I14.70-75

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 DOI : 10.36893.DRSR.2021.V08I14.70-75