Analysis of Linear Systems with Mittag-Leffler Function type delay

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ABSTARCT

This work explores the possibility of modeling of delay time/dead time in physical systems using the well-known Mittag-Leffler Function (MLF). The MLF is a key function used in the analysis of fractional order systems. Delay or dead time is inherently present in all natural and manmade processes/ systems. This delay is very detrimental the stability and performance of a system. The present work proposes to mathematically model this delay using MLF and presents a detailed analysis of such systems. It is shown that the parameter α R+ plays a crucial role on both time and frequency domain analysis of the systems.

Keywords: Fractional Calculus, Dead-Time System, Mittag-Leffler Function

1. INTRODUCTION

1.1 Fractional Calculus

Study of fractional differential equations has been considerably progressed in recent years that imply importance and place of the fractional calculus in the sciences and engineering. On the other side, according to extensive applications of fractional calculus in natural phenomena like chemical physics, electrical networks, viscoelasticity, porous media, electrical networks, it got many scholar's attention, see articles like [1,2,3,4]. Fractional order system model represents the plant more adequately than integer order model. Fractional order controller is naturally the suitable choice for these fractional order models as well as it is widely used for integer order model also. The significance of fractional order control is that it is a generalization of classical control theory. Most of the works in fractional order control systems are in theoretical nature and controller design and implementation in practice is very small. Many research and studies on real systems in the field of system identification during last few decades have revealed the systems inherent fractional order dynamic behavior. Hence, using the notion of fractional-order, it may be a step closer to the real world life as the real processes are generally or most likely fractional. This is the reason that the real dynamic systems can be better represented by the non-integer dynamic models. However, for many of them, the fractionality may be very small.

The significance of fractional order control system is that it is a generalization of classical control theory which could lead to more adequate modeling and more robust control performance. Despite of this fact, the integer-order controls are still more welcome due to absence of accurate solution methods for fractional order differential equations (FODEs). But recently, many progresses in the analysis of dynamic system modeled by FODEs have been made and approximation of fractional derivatives and integrals can be used in the wide area of fractional order control systems. It is also observed that PID controllers which have been modified using the notion of fractional order integrator and differentiator applied to the integer order or fractional order plant enhance the system control performance. As discussed earlier, fractional order systems had been applied to model many real-world phenomena in various fields of physics, engineering and economics, such as dielectric polarization [5], electromagnetic waves [6], viscoelastic system [7], heat conduction [8], biology [9], finance [10], and control theory [11,12]. The complex derivative D^{a+jb}, with a, b as real numbers, is a generalization of the concept of integer derivative, where a = 1, b = 0. Recently, several studies focused on the application of complex-order derivatives,[13,14,15] which modeled and analyzed control theory,[16] chaos,[17] and elements[18] with it.

Fractional calculus study is almost as old as Integral Calculus. Many mathematical theories that are applied to the fractional calculus study were developed prior to the turn of the 20th century. Most fascinating bounds in the field of engineering and scientific application have been found in the past 100 years. To meet the requirements of physical reality, some cases of mathematics had changed. Reformulation of standard definition of the Riemann-Liouville fractional derivative was done by Caputo which lead to the use of integer order initial conditions to solve his fractional order differential equations [2]. Kolowankar again reformulated the Riemann-Liouville fractional derivative in order to differentiate no-where differentiable fractal functions in 1996 [2]. Leibniz's response, based on studies over the intervening 300 years, has proven at least half right. It is clear that within the 20th century especially numerous applications and physical manifestations of fractional calculus have been found. However, these applications and the mathematical background surrounding fractional calculus are far from paradoxical.

Differintegral operator's wide domain of applications is one of its advantages. Within mathematics, the subject is in contact with a very large segment of classical analysis and provides a unifying theme for a great number of well known, and some new results [6]. Except mathematics other application areas includes transmission line theory, chemical analysis of aqueous solutions, design of heat-flux meters, rheology of soils, growth of inter- granular grooves on metal surfaces, quantum mechanical calculations, and dissemination. A recent extension of fractional order system is the complex order (CO) systems. The basic underlying mathematical tool for this is the definitions of complex order derivatives and integrals, where the order is a complex number z Ɛ c, that is, z = a + jb with a, b E R. Thus, the order of the differentiation and integrals are already generalized to incorporate complex order powers. This is true because Γ (.) is also defined for complex arguments. Furthermore, these derivatives and integrals, being linear operators are Laplace transformable. So it is possible to analyze the solution of linear complex order differential equations. Thus, the necessary mathematical framework for the analysis of complex order systems is already available. Already, it has been shown in the literature that fractional order description of the real-world and engineering system provide a more faithful representation. This is due to the infinite-dimensional nature of fractional order system. So it can be very well guaranteed that differential equation with complex order integro-differential operators will surely provide an even better representation of these systems. Following are the possible ways of research/analysis in this new field:

1. Analysis of complex order linear and nonlinear systems.
2. Design of Complex order controllers for the system.

This work focuses on the frequency domain analysis of linear complex order system.

1.1.1 Riemann-Liouville fractional derivative and integral

A commonly used definition of the fractional differintegral is called the Riemann-Liouville definition given as follows [20]:

$$aD_t^\alpha f(t) = \frac{1}{\Gamma(m-a)}\left(\frac{d}{dt}\right)^m \int_\alpha^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}}dt \quad \text{.......... (1.1)}$$

The classical form of fractional calculus is given by the Riemann–Liouville integral, which is essentially what has been described above. The theory for periodic functions (therefore including the 'boundary condition' of repeating after a period) is the Weyl integral. It is defined on Fourier series, and requires the constant Fourier coefficient to vanish (thus, it applies to functions on the unit circle whose integrals evaluate to 0).

$$aD_t^\alpha f(t) = \lim_{h=0} \frac{1}{\Gamma(\alpha)h^\alpha} \sum_{k=0}^{\frac{(t-a)}{h}} \frac{\Gamma(\alpha+K)}{\Gamma(k+1)} f(t-kh) \quad \text{.......... (1.2)}$$

$$aD_t^\alpha f(t) = \frac{1}{\Gamma(m-a)}\left(\frac{d}{dt}\right)^m \int_\alpha^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}}dt \quad \text{.......... (1.3)}$$

for (m 1 < α < m) where Γ(Δ) is the well-known Euler's gamma function.

It was touched upon in the introduction that the formulation of the concept for fractional integrals and derivatives was a natural outgrowth of integer order integrals and derivatives in much the same way that the fractional exponent follows from the more traditional integer order exponent. For the latter, it is the notation that makes the jump seems obvious. While one cannot imagine the multiplication of a quantity a fractional number of times, there seems no practical restriction to placing a non-integer into the exponential position. Similarly, the common formulation for the fractional integral can be derived directly from a traditional expression of the repeated integration of a function. This approach is commonly referred to as the Riemann-Liouville approach.

1.1.2 Grunwald-Letnikov fractional derivative

An alternative approach, based on the concept of fractional-order differentiation, is the Grunwald- Letnikov definition given by

$$aD_t^{\alpha} f(t) = \lim_{h=0} \frac{1}{\Gamma(\alpha) h^{\alpha}} \sum_{k=0}^{\frac{(t-a)}{h}} \frac{\Gamma(\alpha+K)}{\Gamma(k+1)} f(t-kh) \quad \text{.......... (1.4)}$$

In mathematics, the Gru¨nwald–Letnikov derivative is a basic extension of the derivative in fractional calculus that allows one to take the derivative a non-integer number of times. It was introduced by Anton Karl Grunwald (1838–1920) from Prague, in 1867, and by Aleksey Vasilievich Letnikov (1837–1888) in Moscow in 1868.

Before Newton and Leibniz, only algebra, geometry and statistics can be studied. We cannot study dynamics and dynamic systems without (integer-order) calculus. Just like the simple fact that in between integers there are non-integers, today, it is being accepted that fractional order calculus will be more and more useful in various sciences and engineering branches.

Due to the summation form of the Grunwald-Letnikov definition of the fractional derivative and integral, this formula lends itself to adaptation for use in a computer numerical solver. In One question that arises in the programming of such a function is, because the Grunwald- Letnikov definition specifies what is essentially an infinite sum, what number of these terms must be computed and summed for an accurate result to be achieved. Because of the speed of modern computing equipment and the relative simplicity of the step by step calculation (except perhaps the calculation of the factorial), one might suppose that anywhere from 10000 to 1000000 steps would provide excellent accuracy without a significant penalty in computation time. In theory, this would be a correct assumption, as even the calculation of Gamma is done approximately in most mathematical software packages including MATLAB.

1.1.3 Caputo fractional derivative

There is another option for computing fractional derivatives; the Caputo fractional derivative. It was introduced by M. Caputo in his 1967 paper [6]. In contrast to the Riemann Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions. Caputo's definition is illustrated as follows.

$$aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau) d\tau}{(t-\tau)^{\alpha+1-n}} \quad \text{.......... (1.5)}$$

1.2 Mittag-Leffler Function

In 1903, Swedish mathematician Gosta Mittag-Leffler introduced the function Eα(Z). It may be defined by the following series when the real part of α is strictly positive:

$$E_{\alpha}(Z) = \sum_{k=0}^{\infty} \frac{Z^k}{\Gamma(\alpha k + 1)}, \; for \; \alpha > 0, \quad \text{.......... (1.6)}$$

Where Z is complex variable and Γ(.) is Gamma function [7]. The Mittag-Leffler function is a direct generalization of the exponential function to which it reduces for $\alpha = 1$. For $0 < \alpha < 1$, it interpolates between the pure exponential function and hyper geometric function 1 . Its importance is realized during the last two decades due to involvement in the problems of physics, chemistry, biology, engineering and applied sciences. Mittag- Leffler function naturally occurs as the solution of fractional-order differential equation or fractional-order integral equations [4]. Mittag-leffler function (MLF) of two parameters was proposed by Wiman using series expansion and is defined as

$$E_{\alpha,\beta}(Z) = \sum_{k=0}^{\infty} \frac{Z^k}{\Gamma(\alpha k + \beta)}, \; for \; \alpha > 0, \beta > 0, \quad \text{....... (1.7)}$$

where, α and β are two complex parameters and $Re(\alpha) > 0$, $Re(\beta) > 0$ [16], which is known as Wiman's function or generalized Mittag-Leffler function [6]. Later, this function was rediscovered and intensively studied by Agarwal and Humbert. In the case, α and β are real and positive, the series converges for all values of the argument Z, so the Mittag-Leffler function is an entire function. For $\alpha > 0$, the Mittag-Leffler function $E_{\alpha,1}$ is an entire function of order $1/\alpha$, and is in some sense the simplest entire function of its order [7]. A generalized Mittag-Leffler function of four parameters was given by Shukla and Prajapati [6]:

$$E_{\alpha,\beta}^{\gamma,q}(Z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} Z^k}{k!\Gamma(\alpha k+\beta)},\ for\ \alpha>0, \beta>0, \gamma>0, q>0. \quad \text{……. (1.8)}$$

1.3 Time delay systems

The subject of time-delay systems is a rather old research topic dating back to the works of Euler-Bernoulli in the XVIII century. Effective results in this area were initiated in the late fifties of the twentieth century since the works of Krasovskii and Razumikhin on the Lyapunov functions. However the significant progression in this area has been made during the last decade where numerous books, research, and survey papers and special issues have been devoted. Delay is not just a mathematical exercise but more importantly is rooted in many natural and man-made systems such as biology, processes industries, and mechatronic motions. In engineering applications, time-delays generally describe propagation phenomena, material or energy transfer in intercommoned systems and data transmission in communication systems. They have been the main sources inducing oscillations, instability, and poor control performances. Stability analysis and robust control of such systems are then of theoretical and practical importance. Much effort in the analysis and synthesis of these systems has been dedicated to delay-dependent and delay-independent issues based both/either on Lyapunov methods and/or frequency domain techniques. However, it should be noted that there are still numerous challenging issues pending in many classes of time-delay systems.

Time delays are quite frequently encountered in industrial applications, such as heat exchanges, distillation units, mining processes, steel manufacturing and so on. But they are not limited to industrial applications. Time delay processes span from biological to mechanical systems, including also economical or electrical fields. The physical phenomenon that generates time delays is the need to transport information, energy or different masses. Time lags accumulate also between interconnected systems or arise when sensors need measure and acquire signals and when microcontrollers (or other devices) compute the control signal and actuate upon the process.

From the frequency domain point of view, the presence of delay introduces an additional lag in the process phase. This results in lower phase and gain margins and ultimately complicates the closed loop control of these processes. The ideal situation is to design a controller that completely eliminates the effect of time delays. Many control strategies have been developed throughout the years to cope with time delay characteristics [1,4], but none of them proved to be an ideal solution. As the domain of fractional order controller gained more popularity, the control focus also reached the field of time delays processes. The desire is to combine the better performance of fractional calculus to the time delay control problem by extending fractional order design methods to the time delay field. Research output from the delay free processes suggest that using fractional order controllers can help improve robustness and closed loop response of time delay processes as well [4,7].

The paper is organized as follows. Section 1 presents the introduction to fraction calculus, dead-time systems and the ML function definition and application. Section 2 discusses systems with Delay/Dead Time in detail. Section 3 explains ML function and its applications. Section 4 presents the analysis of linear systems with dead-time supported by simulation studies and section 5 concludes the work.

2. SYSTEMS WITH DELAY/DEAD TIME

It's the foremost preliminary step for proceeding with any research work writing. While doing this go through a complete thought Many processes in industry, as well as in other areas, exhibit dead times in their dynamic behavior. In fact, most of the tuning methods for PID controllers used in industry consider dead times as an integral part of process dynamics models. Dead times are mainly caused by information, energy or mass transportation phenomena, but they can also be caused by processing time or by the accumulation of time lags in a number of simple dynamic systems connected in series. For processes exhibiting dead time, every action executed in the manipulated variable of the process will only affect the controlled variable after the process dead time. Because of this, analyzing and designing controllers for dead-time systems are more difficult.

For Example: Dead times are present almost everywhere. Consider, for example, the central heating system of a building. The boiler is usually located in the basement and linked to all rooms by pipes that transport hot water. When the gas valve position is increased, the temperature of the water inside the boiler starts to rise; however, it is necessary to wait a certain amount of time for this heated water to reach the rooms. This dead time is due to the time taken for the hot water to be transported from the central heater to the rooms and depends on the distance and the flow values.

Dead times are also inevitable in communication systems. An example of this is the remote control of vehicles, such as a Lunokhod 1 lunar robot. Lunokhod 1 was designed to operate for 90 days while being guided in real time by a five person team at the Deep Space Center near Moscow, Russia, [43]. It was the first remote-controlled roving robot on the Moon, brought down on the moon by the Soviet Luna 17 spacecraft on 17 November 1970. Lunokhod 1 toured the lunar Mare Imbrium (Sea of Rains) for 11 months and was one of the greatest successes of the Soviet lunar exploration program [73].

When it was necessary to transmit instructions from the earth platform to the mobile robot a dead time composed of the time taken by the signal to travel from the earth to the moon and vice versa had to be taken into account in the communications system. Here, the dead time depends on the velocity of the signal and the distance between the earth and the moon.

2.1 Difficulties in Controlling Dead-time Systems

Processes with significant dead times are difficult to control using standard feedback controllers mainly because of the following: (a) the effect of the disturbances is not felt until a considerable time has elapsed, (b) the effect of the control action takes some time to be felt in the controlled variable and (c) the control action that is applied based on the actual error tries to correct a situation that originated some time before. These difficulties can also be explained in the frequency domain: The dead time introduces an extra decrease in the systems phase, which may cause instability.
A simple example illustrates these difficulties. Consider the central heater with a long dead time previously described. fig(2.2) shows typical behavior of a system of this type, where T is the temperature measured by a sensor located in the room and V represents the gas valve position in the central heater.

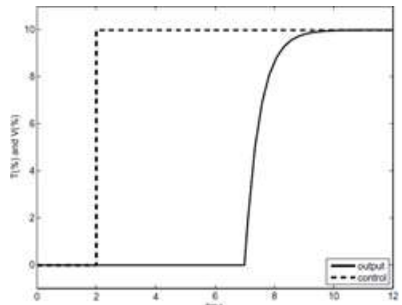


Figure 1: Step response of a dead-time system. Increments in the temperature and the valve position of the simulated process [66].

A proportional + integral (PI) controller can be used to obtain an off-set free closed- loop system. Figure 1 shows the closed-loop behavior of the system when a change of 1% is introduced at the set-point.

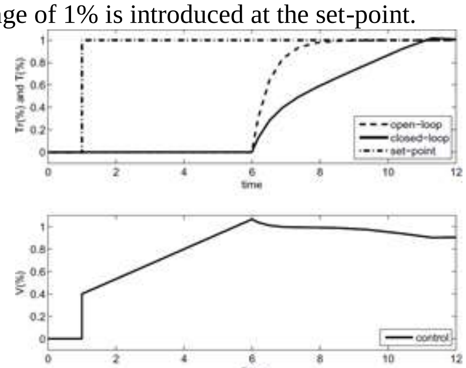


Figure 2: Closed-loop temperature response and control action for the PI controller, reference and open-loop response [66].

A small integral action and controller gain are necessary to obtain no oscillatory behavior in this control system. Note that the settling time in this case is much greater than in the open-loop one. This is the price to be paid for a zero steady-state error and a no oscillatory closed-loop response. If the gain of the controller is increased to obtain a faster closed-loop response, the obtained behavior is very oscillatory. Dead time has two important effects on the closed-loop system. The first one is the physical constraint that does not allow the temperature to react until L periods of time after the change in the valve position. Nothing can be done about this. The second effect is the deterioration of the closed-loop transient after the dead time. Some improvements can be made here. In fact, there has been a significant quantity of research over the last 50 years oriented at controlling dead-time systems.

2.2 Dead-Time Processes

Dead times appear in many processes in industry and in other fields, including economical and biological systems. They are caused by some of the following phenomena: (a) The time needed to transport mass, energy or information; (b) the accumulation of time lags in a great number of low-order systems connected in series; and (c) the required processing time for sensors, such as analyzers; controllers that need some time to implement a complicated control algorithm or process. Dead times introduce an additional lag in the system phase, thereby decreasing the phase and gain margin of the transfer function making the control of these systems more difficult. This chapter gives an introduction to the modeling, analysis and control of dead-time systems. Several ideas will be introduced and will be discussed in the following chapters.

2.2.1 Example: A Heated Tank with a Long Pipe.

Figure 3: A heated tank and a long pipe [66].

Consider a water heater system such as the one shown in Fig. 3, the water is heated in the tank using an electric resistor and driven by a pump along a thermally insulated pipe to the output of the system. The control input is the power W at the resistor and the plant output is the temperature T at the end of the pipe. A linear model of the process can be obtained using a simple step-test identification procedure close to an operation point W0, T0. When a positive step is applied at W, the temperature inside the tank starts to increase. As the pipe is full of water at the initial temperature T0, this change is not immediately perceived at the output and it is necessary to wait until the hot water reaches the end of the pipe before it is noticed. Thus, after a dead time, defined by the flow and the length of the pipe, the output temperature T starts to rise with the same dynamics as the temperature inside the tank. When a constant flow of water F is used, the dead time L can be estimated using F and the volume of the pipe V as, L= V/F.

Figure (4) shows the behavior of T when a step is applied at W. In this simulated situation, the power W (dashed line) changes from 40% to 50% at t = 1 and the temperature (solid line) increases from 55% to 65%. Note that the temperature inside the tank Ti (dotted-dashed line) starts rising at t = 1 s, while the temperature at the end of the pipe only reacts at t = 6 s, thus, there is a dead time of 5 s due to the time needed for mass transportation. Therefore, it is possible to relate the two temperatures Ti (t) = T (t+ 5).

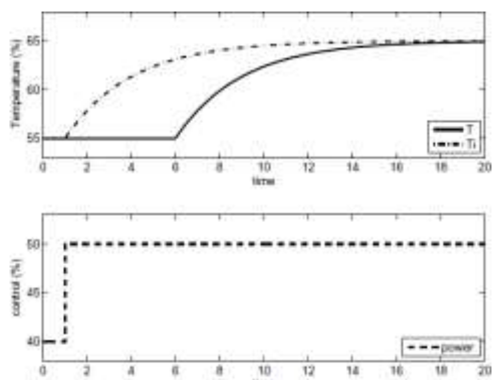


Figure 4: Step response of the system: Ti (dotted-dashed line), T (solid line) and W (dashed line)[66].

Suppose now that a linear model is used to represent the dynamic relationship be- tween the variations on Ti (ΔTi) and the variations on W (ΔW). The transfer function between ΔTi and ΔW is given by,

$$G(s) = \frac{\Delta T_i(s)}{\Delta W(s)} \Rightarrow \Delta T_i(s) = G(s)\Delta W(s).$$

If a generic dead time L is considered, and the Laplace transform is used Lx(t + L) = eLsLx(t), it follows that.

$$\Delta T_i(s) = e^{Ls}\Delta T(s) \Rightarrow \Delta T(s) = \Delta T_i(s)e(-Ls)$$

Thus,

$$\frac{\Delta T(s)}{\Delta W(s)} = G(s)e^{-Ls} \text{ for } L > 0$$

Which is the linear model most used to represent the behavior of dead-time processes.

3 Mittag-Leffler function

3.1. Introduction

The special function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)},\ \alpha \in C, \Re(\alpha) > 0,\ z \in C\text{.........}3.1$$

And its general form

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)},\ \alpha,\beta\in C,\Re(\alpha)>0,\Re(\beta)>0,z\in C\ldots.3.2$$

with C being the set of complex numbers are called Mittag-Leffler functions. The former was introduced by Mittag-Leffler[2,3], in connection with his method of summation of some divergent series. In his papers [2-4], he investigated certain properties of this function. Function defined by (3.1) first appeared in the work of Wiman [5,6]. The function (3.1) is studied, among others, by Wiman[5,6], Agarwal [7], Humbert [8], and Humbert and Agarwal [9] and others. The main properties of these functions are given in the book by Erdelyi et al. [1, Section 18.1], and a more comprehensive and a detailed account of Mittag-Leffler functions is presented in Dzherbashyan [10, Chapter 2]. In particular, functions (3.1) and (3.2) are entire functions of order $\rho=1/\alpha$ and type$\sigma=1$; see, for example, [1, page 118].

The Mittag-Leffler function arises naturally in the solution of fractional order integral equations or fractional order differential equations, and especially in the investigations of the fractional generalization of the kinetic equation, random walks, Lévy flights, superdiffusive transport and in the study of complex systems. The Mittag-Leffler function is not given in the tables of Laplace transforms, where it naturally occurs in the derivation of the inverse Laplace transform of the functions of the type $p^{\alpha}(a+bp^{\beta})$, where p is the Laplace transform parameter and a and b are constants. This function also occurs in the solution of certain boundary value problems involving fractional integro-differential equations of Volterra type [16]. During the various developments of fractional calculus in the last four decades this function has gained importance and popularity on account of its vast applications in the fields of science and engineering. During the last 15 years the interest in MittagLeffler function and Mittag-Leffler type functions is considerably increased among engineers and scientists due to their vast potential of applications in several applied problems, such as fluid flow, rheology, and diffusive transport akin to diffusion, electric networks, probability, and statistical distribution theory.

3.2 Functional Relations for the Mittag-Leffler Functions

In this section, we discuss the Mittag-Leffler functions of rational order $\alpha=m/\ n$, with $m,n\in N$ relatively prime. The differential and other properties of these functions are described in Erdélyi et al. [1] and Dzherbashyan [10].

Theorem 3.1 The following results hold:

$$\frac{d^m}{dz^m}E_m(z^m)=E_m(z^m),$$

$$\frac{d^m}{dz^m}E_{m/n}\left(z^{m/n}\right)=E_{m/n}\left(z^{m/n}\right)+\sum_{r=1}^{n-1}\frac{z^{-rm/n}}{\Gamma(1-rm/n)},\ n=2,3,\ldots,$$

$$E_{m/n}(z)=\frac{1}{m}\sum_{r=1}^{m-1}E_{1/n}\left(z^{1/m}\exp\left(\frac{i2\pi r}{m}\right)\right),$$

$$E_{1/n}\left(z^{1/n}\right)=e^z\left[1+\sum_{r=1}^{n-1}\frac{r(1-r/n,z)}{\Gamma(1-r/n)}\right],\ n=2,3,\ldots,$$

Where $\gamma(a,z)$ denotes the incomplete gamma function, defined by $\gamma(a,z)=\int_0^z e^{-t}t^{a-1}\,dt$.

We also recall the identity $\sum_{r=0}^{m-1}\exp\left[\frac{i2\pi kr}{m}\right]=\begin{cases}m & \text{if } k=0(\mathrm{mod}m)\\ 0 & \text{if } k\neq 0(\mathrm{mod}m)\end{cases}$

we find that $\sum_{r=0}^{m-1}E_{\alpha}\left(ze^{i2\pi r/m}\right)=mE_{\alpha m}(z^m),\ m\in N$

This can be written as, $E_{\alpha}(z)=\frac{1}{m}\sum_{r=0}^{m-1}E_{\alpha/m}\left(z^{1/m}e^{i2\pi r/m}\right),\ m\in N$ and above result now follows by taking $\alpha=m/n$.

To prove above relation we set $m=1$ in and multiply it by $\exp(-z)$ to obtain $\frac{d}{dz}\left[e^{-z}E_{1/n}\left(z^{1/n}\right)\right]=e^{-z}\sum_{r=1}^{m-1}\frac{z^{-r/m}}{\Gamma(1-r/m)}$

On integrating both sides of the above equation with respect to z and using the definition of incomplete gamma function (2.15), we obtain the desired result (2.14). An interesting case of (2.18) is given by

$$E_{2\alpha}(z^2)=\frac{1}{2}[E_{\alpha}(z)+E_{\alpha}(-z)].$$

The Mittag-Leffler function has been introduced to give an answer to a classical question of complex analysis, namely to describe the procedure of the analytic continuation of power series outside the disc of their convergence. The Mittag-Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag-Leffler function plays an analogous role in the solution of non-integer order differential equations. In fact, the exponential function itself is a very specific form, one of an infinite set, of this seemingly ubiquitous function. The importance of the Mittag-Leffler function was re-discovered when its connection to fractional calculus was fully understood [10]. During the last two decades this function has come into

prominence after about nine decades of its discovery by a Swedish Mathematician Mittag-Leffler, due to the vast potential of its applications in solving the problems of physical, biological, engineering, and earth sciences [18].
The Mittag-Leffler function arises naturally in the solution of fractional-order integral equations or fractional-order differential equations, and especially in the investigations of the fractional generalization of the kinetic equation, random walks, Levy flights, superdiffusive transport and in the study of complex systems[13,20]. The ordinary and generalized Mittag- Leffler functions interpolate between a purely exponential law and power-law-like behavior of phenomena governed by ordinary kinetic equations and their fractional counterparts [10].

3.3. Applications of Mittag-Leffler Functions in Fractional Order Modeling

The branch of fractional modeling can be formally divided into two sub directions, namely the creation and study of deterministic models and stochastic models [1]. The first sub direction is more developed. Anyway, it should be noted that there is a large amount of articles where the models are introduced formally. Among the models dealing with more practical applications, we can point out those related to fractional viscoelasticity [15]. Linear viscoelasticity is certainly the field of the most extensive applications of fractional calculus, in view of its ability to model hereditary phenomena with long memory[49]. During the twentieth century, a number of authors have (implicitly or explicitly) used the fractional calculus as an empirical method for describing the properties of viscoelastic materials. Such investigations are based on the classical works by Scott-Blair, Gemant, Gerasimov and Rabotnov [15,17,50]. The beginning of the modern applications of fractional calculus in linear viscoelasticity is generally attributed to the 1979 Ph.D. thesis by Bagley (under supervision of Prof. Torvik), followed by a number of relevant papers. However, for the sake of completeness, one would recall also the 1970PhD thesis of Rossikhin under the supervision of Prof. Meshkov and the 1971 PhD thesis of Mainardi under the supervision of Prof. Caputo.
As applications in physics and chemistry, we would like to quote the contributions by Kenneth S. Cole (1933), quoted in connection with nerve conduction, and by de Oliveira Castro (1939), Kenneth S. Cole and Robert H. Cole (1941-1942) and Gross (1947) in connection with dielectric and mechanical relaxation, respectively [18,51]. Subsequently, in 1971, Caputo and Mainardi [15] proved that the Mittag-Leffler function is present whenever derivatives of fractional order are introduced into the constitutive equations of a linear viscoelastic body. Since then, several other authors have pointed out the relevance of the Mittag-Leffler function to fractional viscoelastic models [15,52 − 54] and the references therein). We also mention here a growing number of applications of the Mittag-Leffler function in control theory[11,13,16].
Stochastic modeling, which uses the fractional calculus approach, as well as the machinery of the Mittag-Leffler functions, is connected mainly with the concept of the continuous time random walk (CTRW) (see [1,12]). A fractional generalization of the Poisson probability distribution was presented by Pillai in 1990 in his pioneering work. He introduced the probability distribution (which he called the Mittag-Leffler distribution) using the complete monotonicity of the Mittag-Leffler function. The concept of a geometrically infinitely-divisible distribution was introduced in 1984 by Klebanov, Maniya and Melamed. Later, in 1995, Pillai introduced a discrete analogue of such a distribution (i.e., the discrete Mittag-Leffler distribution). Another possible variant of the generalizations of the Poisson distribution is that introduced by Lamperti in 1958, whose density has the expression:

$$f_{x_\alpha}(y) = \frac{\sin \pi\alpha}{\pi} \frac{y^{\alpha-1}}{y^{2\alpha} + 2y^{\alpha}\cos \pi\alpha + 1}$$

The concept of renewal process has been developed as a stochastic model for describing the class of counting processes for which the times between successive events are independent identically distributed (i.i.d.) non-negative random variables, obeying a given probability law. These times are referred to as waiting times or inter-arrival times. The process of the accumulation of waiting times is inverse to the counting number process, called the Erlang process.
The Mittag-Leffler function appears also in the solution of the fractional master equation. Such an equation characterizes the renewal processes with reward modeling by the random walk model known as continuous time random walk. In that, the waiting time is assumed to be a continuous random variable. The name CTRW became popular in physics after publication in the 1960s by Montroll, Weiss and Scher, the celebrated series of papers on random walks to model diffusion processes on lattices. The basic role of the Mittag-Leffler waiting time probability density in time fractional continuous time random walk became well known since the fundamental paper by Hilfer and Anton (1995). Earlier, this conception was used in the theory of thinning (rarefaction) of a renewal process under power law assumptions. CTRWs are rather good and general phenomenological models for diffusion, including anomalous diffusion, provided that the resting time of the walker is much greater than the time it takes to make a jump. In fact, in the formalism, jumps are instantaneous. In more recent times, CTRWs were applied back to economics and finance.
It should be noted, however, that the idea of combining a stochastic process for waiting times between two consecutive events and another stochastic process, which associates a reward or a claim to each event, dates back at least to the first half of the twentieth century with the so-called Cramér-Lundberg model for insurance risk. In a probabilistic framework, we now

find it more appropriate to refer to all of these processes as compound renewal processes. An alternative renewal process called the Wright process (close to the Mittag-Leffler process) was investigated by Mainardi et al.[55] as a process arising by discretization of the stable subordinator. This approach is based on the concept of the extremal Lévy stable density (Lévy stable processes are widely discussed in several books on probability theory). For the study of the Wright processes, an essential role is played by the so-called M-Wright function, a variant of the Mittag-Leffler function [15]. A scaled version of this process has been used by Barkai (2002) for approximating the timefractional diffusion process directly by a random walk subordinate to it (executing this scaled version in natural time), and he has found rather poor convergence in refinement.

4. ANALYSIS OF MITTAG-LEFFLER FUNCTION TYPE DELAY

4.1 Linear system with MLF Delay

Using the laplace transform property of second shifting as given in [67], we define the linear system with MLF delays as

$$\frac{y(s)}{u(s)} = G(s) = \tilde{G(s)}\Big) E_\alpha(-s^\alpha T^\alpha) \qquad \text{---------(4.1)}$$

Where $G(s)$ is any transfer function and m,ay IO,FO, or CO. The delay $E_\alpha(-s^\alpha T^\alpha)$ is an MLF defined as,

$$E_\alpha(-s^\alpha T^\alpha) = \sum_{k=0}^{N} \frac{\left((-1)^k s^\alpha T^\alpha\right)^k}{\Gamma(\alpha k+1)} \qquad \text{----------(4.2)}$$

Where, $\alpha\varepsilon R^+$and T is the delay or dead-time in seconds. It should be noted that for $\alpha = 1, (4.2)$ reduces to the conventional delay system as,

$$E_1(-sT) = e^{-sT}$$

The Physical interpretation of MLF type delay can be given as follows.

1. MLF type delay represents a memory type delay owing to the infinite dimensional property of FO systems. Thus, the amount and nature of delay depends on the value ofα.
2. In conventional delay systems, the distributed delay in the physical system is lumped at the input side. This makes the model to lose important dynamics of the system. The MLF type delay preserves the distributed nature of the delay thereby providing a model which is more realistic.

Thus, the use of MLF type delay is justified and this area is worth exploring for possible new outcomes and results.

4.2 Analysis of systems with MLF delay.

We start with a simple first order stable with MLF delay. Thus,

$$G(\tilde{s}) = \frac{b}{s+a} \qquad \text{---------(4.3)}$$

So, Equation (4.1) becomes,

$$G(s) = \frac{b}{s+a} E_\alpha(-s^\alpha T^\alpha) \qquad \text{----------(4.4)}$$

For the sake of simplicity and to study the effect of Fractional Order α, we assume $a = b = T = 1$. Thus,

$$G(s) = \frac{1}{s+1} E_\alpha(-s^\alpha) \qquad \text{----------(4.5)}$$

The MLF delay is expressed as,

$$E_\alpha(-s^\alpha) = \sum_{k=0}^{N} \frac{(-1)^k s^{\alpha k}}{\Gamma(\alpha k+1)} \qquad \text{-----------(4.6)}$$

The upper limit of the Summation is replaced by $N \in Z^+$(Positive Integer)

Now let us analyze system 4.6, For $N = 1\&\ N = 2$.

4.3 Analysis for N = 1

$$E_\alpha(-s^\alpha) = \sum_{k=0}^{1} \frac{(-1)^k s^{\alpha k}}{\Gamma(\alpha k + 1)}$$

$$= \left[\left[1 - \frac{s^\alpha}{\Gamma(\alpha + 1)}\right]\right]$$

Substituting in equation (4.5) gives,

$$G(s) = \frac{1}{s+1}\left[1 - \frac{s^\alpha}{\Gamma(\alpha+1)}\right]$$

$$G(s) = \frac{\Gamma(\alpha+1) - s^\alpha}{\Gamma(\alpha+1)[s+1]} \qquad \text{-----(4.7)}$$

Thus, $Y(s) = G(s)U(s)$,

Considering unit step input with$U(s) = \frac{1}{s}$, the output is given by,

$$Y(s) = \frac{\Gamma(\alpha+1) - s^{\alpha}}{\Gamma(\alpha+1)[s+1]} \frac{1}{s}$$

Simplifying

$$y(s) = \frac{1}{s(s+1)} - \frac{1}{\Gamma(\alpha+1)s^{1-\alpha}(s+1)}$$

Thus,

$$y(t) = \pounds^{-1}\left[\frac{1}{s(s+1)}\right] - \pounds^{-1}\left[\frac{1}{\Gamma(\alpha+1)s^{1-\alpha}(s+1)}\right]$$

The inverse Laplace T_r of second term is obtained using the following identity give in [68].

$$y(t) = \pounds^{-1}\left[\frac{1}{s(s+1)}\right] = t^{\alpha}E_{1,1+\alpha}(at)$$

Thus,

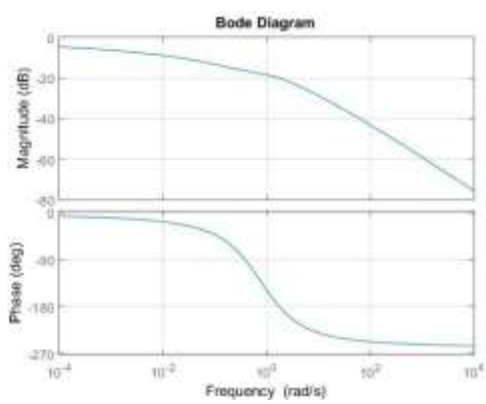
$$Y(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{\Gamma(\alpha+1)} \frac{1}{s^{1-\alpha}(s+1)}$$

The Time-domain expression for step response is given as:

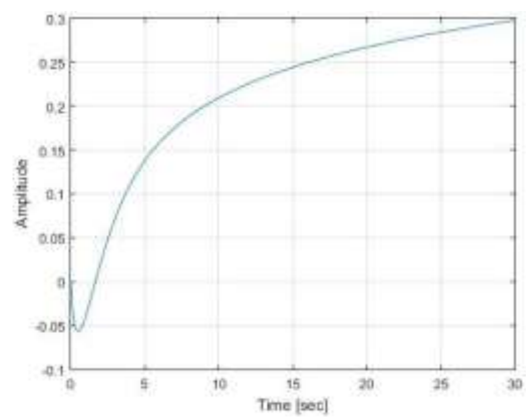
$$y(t) = 1 - e^{-t} - \left[\frac{1}{\gamma(\alpha+1)}\right] t^{1-\alpha}E_{1,2-\alpha}(-t)$$

A closer look at (4.8) reveals that the output $y(t)$ is bounded implying that the system (4.7) is stable.
We now use the MATLAB toolbox FOMCON [69] to evaluate the step impulse and bode plots for (4.7). This procedure is also repeated for $N = 2$.

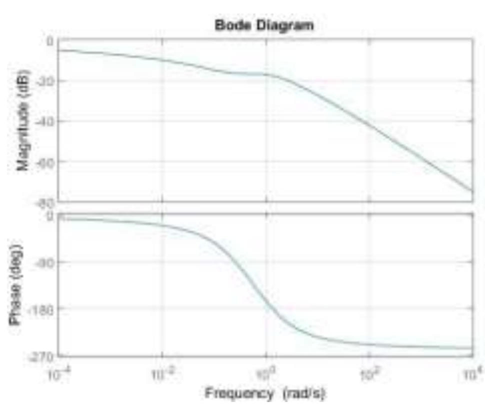
4.4 Results & Analysis for $N = 2$

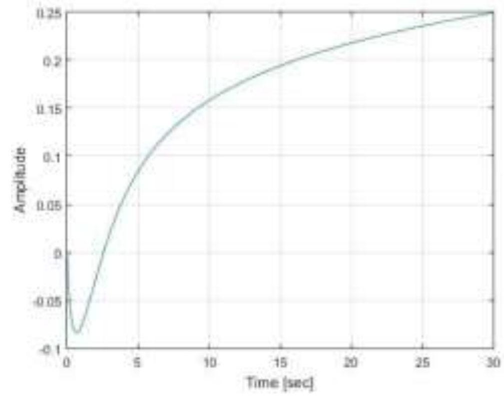


(a) Bode plot for $N = 1$ $\alpha = 0.1$.



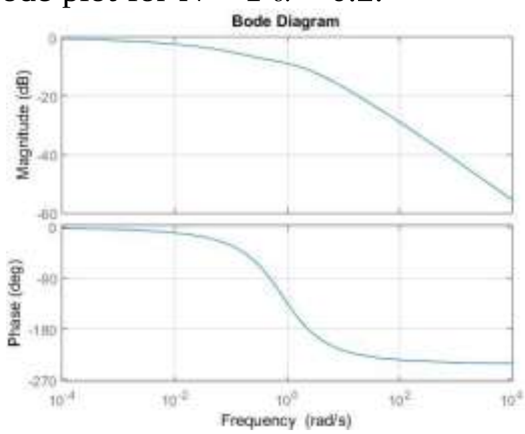
(b) Step response for $N = 1$ $\alpha = 0.1$.



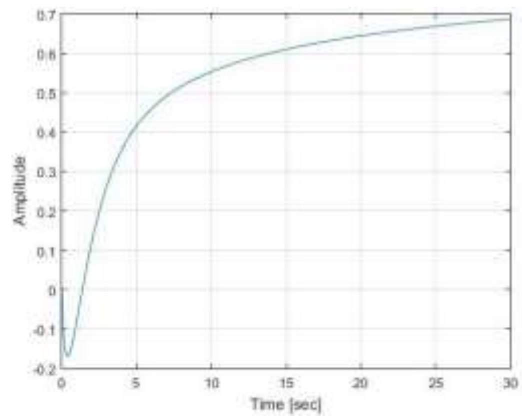


(a) Bode plot for N = 1 α = 0.2.

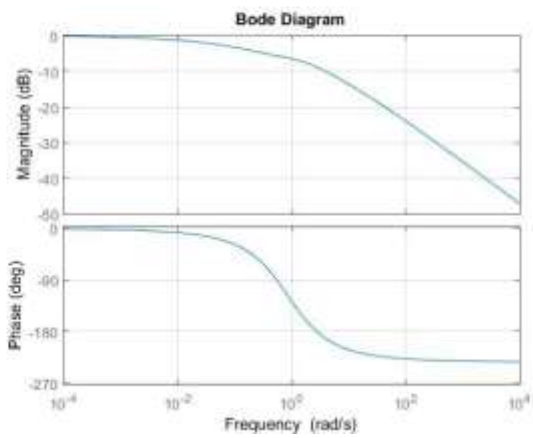
(b) Step response for N = 1 α = 0.2.



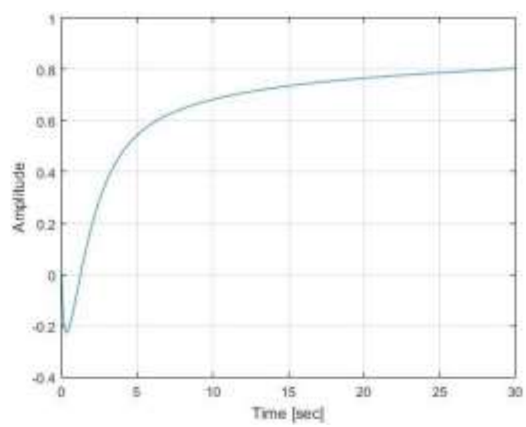
(a) Bode plot for N = 1 α = 0.3.



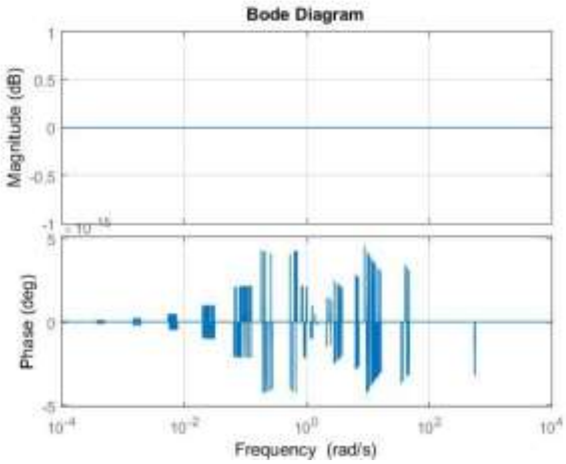
(b) Step response for N = 1 α = 0.3.

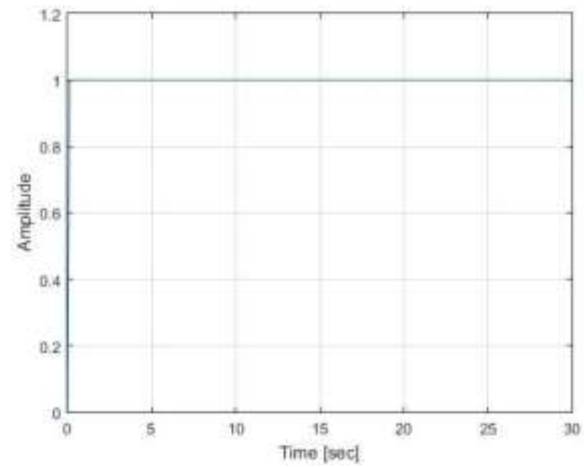


(a) Bode plot for N = 1 α = 0.4.



(b) Step response for N = 1 α = 0.4.





(a) Bode plot for N = 1 α = 0.5. (b) Step response for N = 1 α = 0.5.

Figure 5: Time-and frequency domain plots for N=1.

				N=1					
Alpha	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
Rise Time	323.32	167.95	1336.83	2462.28	0.80	4167.08	3581.58	3033.71	2568.77
Settling Time	4551.65	3820.52	5506.68	6546.18	1.98	7901.17	7097.35	6029.64	4888.51
Settling Minimum	0.72	0.73	0.80	0.79	1.00	0.78	0.83	0.86	0.89
Settling Maximum	0.80	0.82	0.89	0.88	1.00	0.86	0.92	0.96	0.98
Overshoot	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Undershoot	0.00	0.00	18.98	28.24	0.00	45.65	52.15	60.86	73.60
Peak	0.80	0.82	0.89	0.88	1.00	0.86	0.92	0.96	0.98
Peak time	10000	10000	10000	10000	8833	10000	10000	10000	10000
Gain Margin	10.13	7.66	3.58	2.78	Infinite	1.92	1.68	1.47	1.27
Phase Margin	Infinite	Infinite	Infinite	Infinite	Infinite	Infinite	Infinite	Infinite	Infinite

Table 4.1: Result Analysis for N=1 and different values of alpha.

4.5 Results & Analysis for N = 2

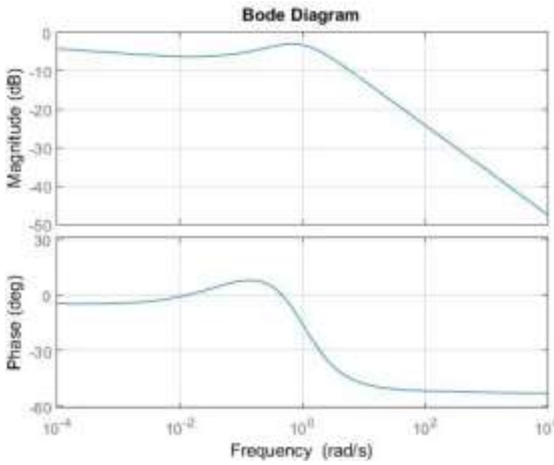
$$\square = \square\tilde{(\square)} \sum_{\square=\square}^{\square} \frac{((-\square)^{\square}\square^{\square}\square^{\square})^{\square}}{\square(\square\square+\square)}$$

$$\square(\square) = \square\tilde{(\square)}\left[\square - \frac{\square^{\square}\square^{\square}}{\square(\square+\square)} + \frac{\square^{\square\square}\square^{\square\square}}{\square(\square\square+\square)}\right]$$

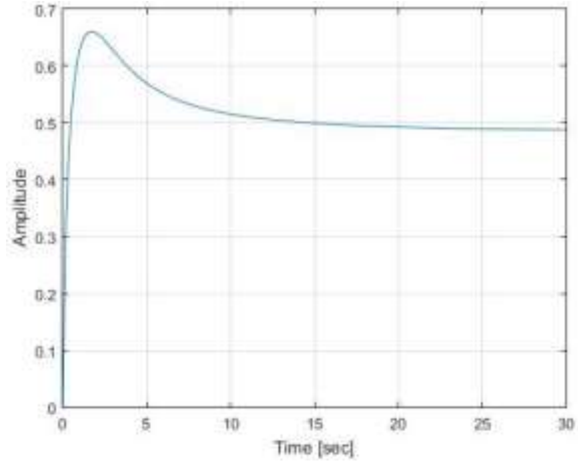
So, from (4.3) $G(\tilde{s}) = \frac{b}{s+a}$, with b = a = 1 & let T = 1. We obtained the transfer function for N = 2 case as follows,

$$\square(\square) = \frac{\square}{\square+\square}\left[\frac{\square(\square+\square)\square(\square+\square\square) - \square(\square\square+\square)\square^{\square} + \square(\square+\square)\square^{\square\square}}{\square(\square+\square)\square(\square\square+\square)}\right]$$

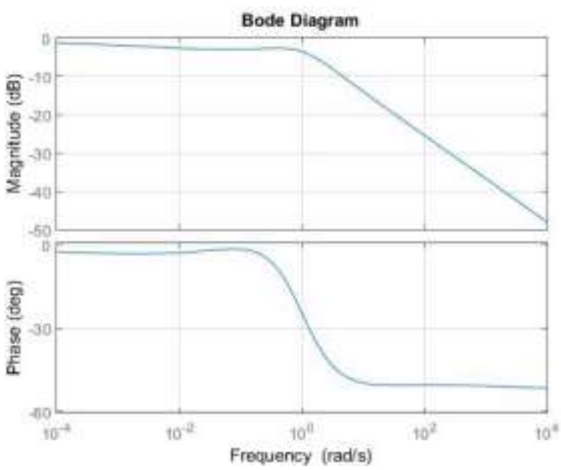
Simulation Results for N = 2 :



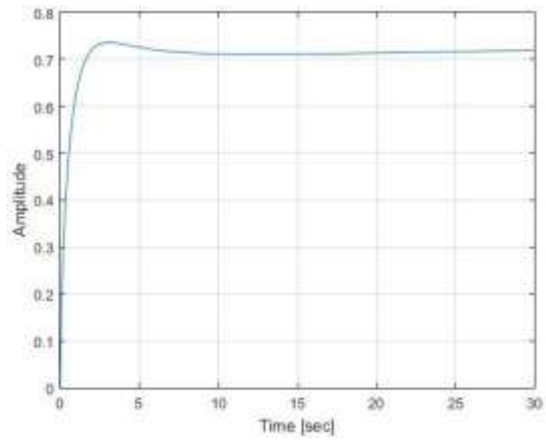
(a) Bode plot for N = 2 α = 0.1.



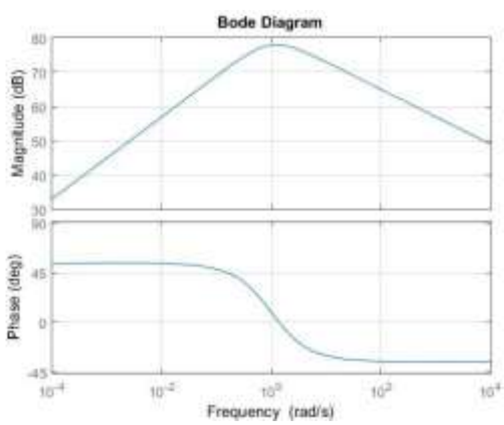
(b) Step response for N = 2 α = 0.1.



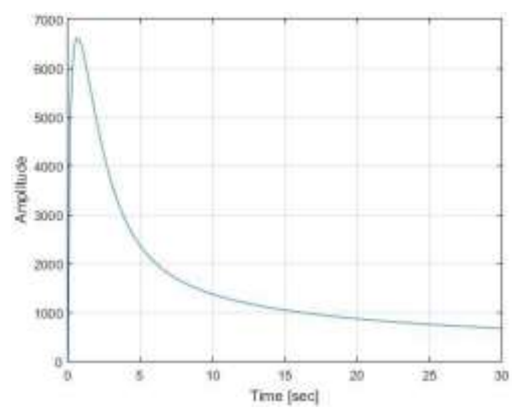
(a) Bode plot for N = 2 α = 0.2.



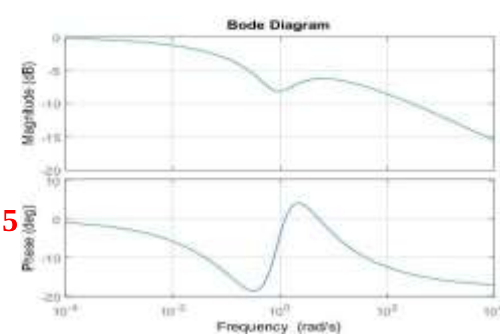
(b) Step response for N = 2 α = 0.2.

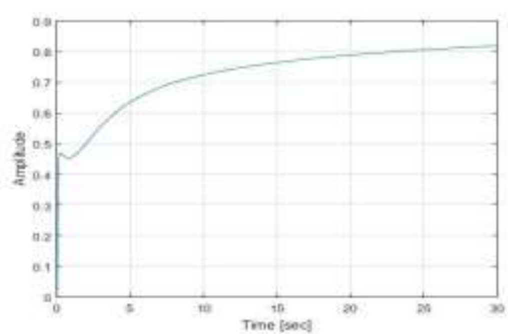


(a) Bode plot for N = 2 α = 0.3.

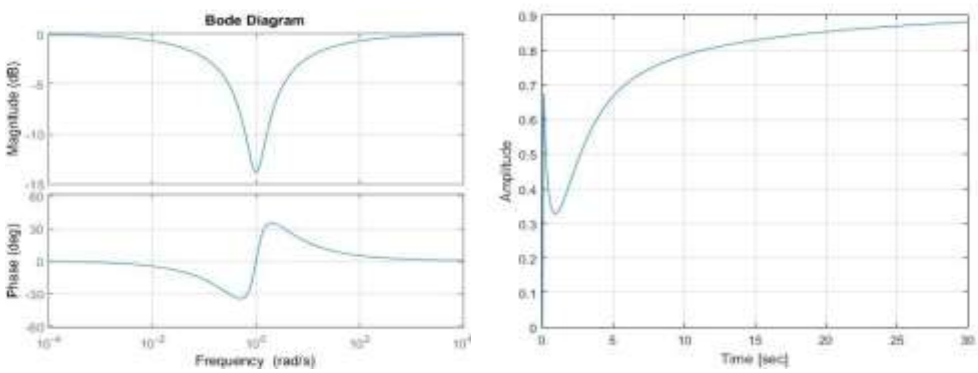


(b) Step response for N = 2 α = 0.3.





(a) Bode plot for N = 2 α = 0.4. (b) Step response for N = 2 α = 0.4.



(a) Bode plot for N = 2 α = 0.5 (b) Step response for N = 2 α = 0.5.

Figure 6: Time-and frequency domain plots for N=2.

N=2					
Alpha	0.1	0.2	0.3	0.4	0.5
Rise Time	102.08	23.69	2102.55	2214.79	0.77
Settling Time	4138.09	5639.04	7098.07	6865.49	2733.01
Settling Minimum	0.68	0.71	0.73	0.81	0.96
Settling Maximum	0.86	0.81	0.82	0.90	1.00
Overshoot	14.43	0.00	0.00	0.00	4.16
Undershoot	0.00	0.00	0.00	0.00	0.00
Peak	0.86	0.81	0.82	0.90	1.00
Peak time	351	10000	10000	10000	2

4.6 Analysis

It is seen from the above plots and tables that, the fractional-order parameter α greatly affects the transient response of the system even if the dead-time T and the parameter N remain unchanged.

The salient observations are enumerated below:

1. Change in α results in drastic changes in transient response of the system.
2. The time for which output goes negative is also affected by the value of α.
3. The frequency response is also affected by α.

5 Conclusion

The novel concept of modeling dead time or delay in a system using MLF has been discussed in this work. The analysis reveals interesting features of such type of systems. It is seen that the FO parameter α plays an role in the time domain and frequency domain indices of the systems.

REFERENCES

1. D. N. Sonawane, M. S. Sutaone , B. N. Choudhari and Abhijeet Badurkar , "FPGA Implementation of Simplified SVPWM Algorithm for Three Phase Voltage Source Inverter ", International Journal of Computer and Electrical Engineering, Vol.2, No.6, December, pp. 1793-8163, 2010.
2. Kilbas, A.A.A., Srivastava, H.M. and Trujillo, J.J., 2006. Theory and applications of fractional differential equations (Vol. 204). Elsevier Science Limited.
3. Sumelka, W., 2014. Fractional viscoplasticity. Mechanics Research Communications, 56, pp.31-36.
4. Machado, J.T., Galhano, A.M. and Trujillo, J.J., 2013. Science metrics on fractional calculus development since 1966. Fractional Calculus and Applied Analysis, 16(2), pp.479-500.
5. Kuschel, M. and Kalkner, W., 1999. Dielectric response measurements in time and frequency domain of different XLPE homo-and copolymer insulated medium voltage cables. IEE Proceedings-Science, Measurement and Technology, 146(5), pp.243-248.
6. Tarasov, V.E., 2009. Fractional integro-differential equations for electromagnetic waves in dielectric media. Theoretical and Mathematical Physics, 158(3), pp.355-359.
7. Zhang, Y., Tian, Q., Chen, L. and Yang, J.J., 2009. Simulation of a viscoelastic flexible multibody system using absolute nodal coordinate and fractional derivative methods. Multibody System Dynamics, 21(3), pp.281-303.
8. Povstenko, Y.Z., 2004. Fractional heat conduction equation and associated thermal stress. Journal of Thermal Stresses, 28(1), pp.83-102.
9. Magin, R.L., 2010. Fractional calculus models of complex dynamics in biological tis- sues. Computers & Mathematics with Applications, 59(5), pp.1586-1593.
10. Elliott, R.J. and Van Der Hoek, J., 2003. A general fractional white noise theory and applications to finance. Mathematical Finance, 13(2), pp.301-330.
11. Tang, Y., Gao, H., Zhang, W. and Kurths, J., 2015. Leader-following consensus of a class of stochastic delayed multi-agent systems with partial mixed impulses. Auto- matica, 53, pp.346-354.
12. Tang, Y., Gao, H., Zhang, W. and Kurths, J., 2015. Leader-following consensus of a class of stochastic delayed multi-agent systems with partial mixed impulses. Auto- matica, 53, pp.346-354.
13. Pinto, C.M. and Machado, J.T., 2011. Complex order van der Pol oscillator. Nonlin- ear Dynamics, 65(3), pp.247-254.
14. Silva, M.F., Machado, J.T. and Barbosa, R.S., 2006. Complex-order dynamics in hexapod locomotion. Signal processing, 86(10), pp.2785-2793.
15. Si, G., Diao, L., Zhu, J., Lei, Y. and Zhang, Y., 2017. Attempt to generalize fractional-order electric elements to complex-order ones. Chinese Physics B, 26(6), p.060503.
16. Machado, J.T., 2013. Optimal controllers with complex order derivatives. Journal of Optimization Theory and Applications, 156(1), pp.2-12.
17. Zhu, H., Zhou, S. and Zhang, J., 2009. Chaos and synchronization of the fractional- order Chua's system. Chaos, Solitons & Fractals, 39(4), pp.1595-1603.
18. Machado, J.T., 2013. Fractional generalization of memristor and higher order el- ements. Communications in Nonlinear Science and Numerical Simulation, 18(2), pp.264-275.
19. Monje, C.A., Chen, Y., Vinagre, B.M., Xue, D. and Feliu-Batlle, V., 2010. Fractional- order systems and controls: fundamentals and applications. Springer Science & Busi- ness Media.
20. Val´erio, D. and Da Costa, J.S., 2013. An introduction to fractional control (Vol. 91). IET.
21. Love, E.R., 1971. Fractional derivatives of imaginary order. Journal of the London Mathematical Society, 2(2), pp.241-259.
22. Ross, B.E.R.T.R.A.M., Haven, W. and NORTHOVER, F.H., 1978. A USE FOR A DERIVATIVE OF COMPLEX ORDER IN THE FRACTIONAL CALCULUS OR.
23. Nigmatullin, R.R. and Le Mehaute, A., 2005. Is there geometrical/physical meaning of the fractional integral with complex exponent?. Journal of non-crystalline solids, 351(33-36), pp.2888-2899.
24. Hartley, T.T., Lorenzo, C.F. and Adams, J.L., 2005, January. Conjugated-order differintegrals. In ASME 2005 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference (pp. 1597-1602). American Society of Mechanical Engineers.
25. Adams, J.L., 2009. Hankel operators for fractional-order systems (Doctoral disserta- tion, University of Akron).
26. Adams, J.L., Hartley, T.T. and Adams, L.I., 2010. A solution to the fundamental linear complex-order differential equation. Advances in Engineering Software, 41(1), pp.70-74.
27. Adams, J.L., Veillette, R.J. and Hartley, T.T., 2010, July. Conditions for stable and causal conjugate-order systems. In 2010 IEEE International Symposium on Industrial Electronics (pp. 108-113). IEEE.
28. Adams, J.L., Veillette, R.J. and Hartley, T.T., 2012. Conjugate-order systems for signal processing: stability, causality, boundedness, compactness. Signal, Image and Video Processing, 6(3), pp.373-380.
29. Jacob, J.A., Tare, A.V., Vyawahare, V.A. and Pande, V.N., 2016, December. A review of time domain, frequency domain and stability analysis of linear complex-order systems. In 2016 IEEE International WIE Conference on Electrical and Computer Engineering (WIECON-ECE) (pp. 164-169). IEEE.
30. Silva, M.F., Machado, J.T. and Barbosa, R.S., 2006. Complex-order dynamics in hexapod locomotion. Signal processing, 86(10), pp.2785-2793.
31. Pinto, C.M. and Machado, J.T., 2011. Complex order van der Pol oscillator. Nonlin- ear Dynamics, 65(3), pp.247-254.
32. Pinto, C.M. and Machado, J.T., 2012. Complex-order forced van der Pol oscillator. Journal of Vibration and Control, 18(14), pp.2201-2209.
33. Pinto, C.M. and Carvalho, A.R., 2015. Effect of drug-resistance in a fractional complex-order model for HIV infection. IFAC-PapersOnLine, 48(1), pp.188-189.
34. Pinto, C.M. and Carvalho, A.R., 2016. Fractional complex-order model for HIV in- fection with drug resistance during therapy. Journal of Vibration and Control, 22(9), pp.2222-2239.
35. Pinto, C.M. and Machado, J.T., 2011. Complex order biped rhythms. International Journal of Bifurcation and Chaos, 21(10), pp.3053-3061.
36. Atanackovic, T.M., Janev, M., Konjik, S., Pilipovic, S. and Zorica, D., 2015. Vibra- tions of an elastic rod on a viscoelastic foundation of complex fractional Kelvin–Voigt type. Meccanica, 50(7), pp.1679-1692.

37. Atanacković, T.M., Konjik, S., Pilipović, S. and Zorica, D., 2016. Complex order frac- tional derivatives in viscoelasticity. Mechanics of Time-Dependent Materials, 20(2), pp.175-195.
38. Adams, J.L., 2009. Hankel operators for fractional-order systems (Doctoral disserta- tion, University of Akron).
39. Machado, J.T., 2013. Fractional generalization of memristor and higher order el- ements. Communications in Nonlinear Science and Numerical Simulation, 18(2), pp.264-275.
40. Si, G., Diao, L., Zhu, J., Lei, Y. and Zhang, Y., 2017. Attempt to generalize fractional-order electric elements to complex-order ones. Chinese Physics B, 26(6), p.060503.
41. Si, G., Diao, L., Zhu, J., Lei, Y. and Zhang, Y., 2017. Attempt to generalize fractional-order electric elements to complex-order ones. Chinese Physics B, 26(6), p.060503.
42. Das, S., Saha, S., Das, S. and Gupta, A., 2011. On the selection of tuning methodol- ogy of FOPID controllers for the control of higher order processes. ISA transactions, 50(3), pp.376-388.
43. Shahiri, M., Ranjbar, A., Karami, M.R. and Ghaderi, R., 2015. Robust control of nonlinear PEMFC against uncertainty using fractional complex order control. Nonlin- ear Dynamics, 80(4), pp.1785-1800.
44. Shah, P. and Agashe, S., 2016. Review of fractional PID controller. Mechatronics, 38, pp.29-41.
45. Kumar, M.R., Deepak, V. and Ghosh, S., 2017. Fractional-order controller design in frequency domain using an improved nonlinear adaptive seeker optimization algorithm. Turkish Journal of Electrical Engineering & Computer Sciences, 25(5), pp.4299-4310.
46. Kumar, M.R., Ghosh, S. and Das, S., 2017. Charge-discharge energy efficiency anal- ysis of ultracapacitor with fractional-order dynamics using hybrid optimization and its experimental validation. AEU-International Journal of Electronics and Communi- cations, 78, pp.274-280.
47. Hartley, T.T., Veillette, R.J., Adams, J.L. and Lorenzo, C.F., 2015. Energy storage and loss in fractional-order circuit elements. IET Circuits, Devices & Systems, 9(3), pp.227-235.
48. Dwivedi, P., Pandey, S. and Junghare, A., 2017. Performance analysis and experimen- tal validation of 2-DOF fractional-order controller for underactuated rotary inverted pendulum. Arabian Journal for Science and Engineering, 42(12), pp.5121-5145.
49. Qiu, J., Tian, H., Lu, Q. and Gao, H., 2013. Nonsynchronized robust filtering design for continuous-time T–S fuzzy affine dynamic systems based on piecewise Lyapunov functions. IEEE transactions on cybernetics, 43(6), pp.1755-1766.
50. Qiu, J., Ding, S.X., Gao, H. and Yin, S., 2015. Fuzzy-Model-Based Reliable Static Output Feedback Control of Nonlinear Hyperbolic PDE Systems. IEEE Transactions on Fuzzy Systems, 24(2), pp.388-400.
51. Qiu, J., Wei, Y. and Wu, L., 2016. A novel approach to reliable control of piecewise affine systems with actuator faults. IEEE Transactions on Circuits and Systems II: Express Briefs, 64(8), pp.957-961.
52. Gawthrop, P.J. and Wang, L., 2006. Intermittent predictive control of an inverted pendulum. Control Engineering Practice, 14(11), pp.1347-1356.
53. Boubaker, O., 2012, July. The inverted pendulum: A fundamental benchmark in control theory and robotics. In International conference on education and e-learning innovations (pp. 1-6). IEEE.
54. Mujumdar, A., Tamhane, B. and Kurode, S., 2015. Observer-based sliding mode control for a class of noncommensurate fractional-order systems. IEEE/ASME Trans- actions on Mechatronics, 20(5), pp.2504-2512.
55. Qiu, J., Wang, T., Yin, S. and Gao, H., 2016. Data-based optimal control for net- worked double-layer industrial processes. IEEE Transactions on Industrial Electronics, 64(5), pp.4179-4186.
56. Qiu, J., Gao, H. and Ding, S.X., 2015. Recent advances on fuzzy-model-based nonlin- ear networked control systems: A survey. IEEE Transactions on Industrial Electronics, 63(2), pp.1207-1217.
57. Si, G., Diao, L., Zhu, J., Lei, Y. and Zhang, Y., 2017. Attempt to generalize fractional-order electric elements to complex-order ones. Chinese Physics B, 26(6), p.060503.
58. Atanackovic, T.M. and Pilipovic, S., 2018. On a constitutive equation of heat conduc- tion with fractional derivatives of complex order. Acta Mechanica, 229(3), pp.1111- 1121.
59. Guefrachi, A., Najar, S., Amairi, M. and Abdelkrim, M., 2012. Frequency response of a fractional complex order transfer function. In 13th International conference on Sciences and Techniques of Automatic control & computer engineering (pp. 765-773).
60. Hanif, O., Babu, G.B. and Sharma, S., 2018, February. Performance Improvement of PI x+ iy D Fractional Complex Order Controller using Genetic Algorithm. In 2018 Fourth International Conference on Advances in Electrical, Electronics, Information, Communication and Bio-Informatics (AEEICB) (pp. 1-5). IEEE.
61. Shahiri, M., Ranjbar, A., Karami, M.R. and Ghaderi, R., 2016. New tuning de- sign schemes of fractional complex-order PI controller. Nonlinear Dynamics, 84(3), pp.1813-1835.
62. Guefrachi, A., Najar, S., Amairi, M. and Aoun, M., 2017. Tuning of fractional com- plex order PID controller. IFAC-PapersOnLine, 50(1), pp.14563-14568.
63. Atanackovic, T.M. and Pilipovic, S., 2018. On a constitutive equation of heat conduc- tion with fractional derivatives of complex order. Acta Mechanica, 229(3), pp.1111- 1121.
64. Sánchez-López, C., Carbajal-Gómez, V.H., Carrasco-Aguilar, M.A. and Carro-Perez, I., 2018. Fractional-order memristor emulator circuits. Complexity, 2018.
65. Normey-Rico, J. and Camacho, E., 2007. Control of dead-time processes. Berlin: Springer.
66. Haubold, H., Mathai, A. and Saxena, R., 2011. Mittag-Leffler Functions and Their Applications. Journal of Applied Mathematics, 2011, pp.1-51.
67. Monje, C.A., Chen, Y., Vinagre, B.M., Xue, D. and Feliu-Batlle, V., 2010. Fractional- order systems and controls: fundamentals and applications. Springer Science & Busi- ness Media.
68. Aleksei Tepljakov (2021). FOMCON Toolbox for MATLAB (https://github.com/extall/fomcon-matlab), GitHub. Retrieved April 1, 2021.